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Upper bounds on the convective heat transport in a rotating fluid layer of infinite Prandtl number: case of large Taylor numbers

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Abstract. By means of the Howard-Busse method of the optimum theory of turbulence we obtain upper bounds on the convective heat transport in a horizontal fluid layer heated from below and rotating about a vertical axis. We consider the interval of large Taylor numbers where the intermediate layers of the optimum fields expand in the direction of the corresponding internal layers. We consider the $1-\alpha$-solution of the arising variational problem for the cases of rigid-stress-free, stress-free, and rigid boundary conditions. For each kind of boundary condition we discuss four cases: two cases where the boundary layers are thinner than the Ekman layers of the optimum field and two cases where the boundary layers are thicker than the Ekman layers. In most cases we use an improved solution of the Euler-Lagrange equations of the variational problem for the intermediate layers of the optimum fields. This solution leads to corrections of the thicknesses of the boundary layers of the optimum fields and to lower upper bounds on the convective heat transport in comparison to the bounds obtained by Chan [J. Fluid Mech. **64**, 477 (1974)] and Hunter and Riahi [J. Fluid Mech. **72**, 433 (1975)]. Compared to the existing experimental data for the case of a fluid layer with rigid boundaries the corresponding upper bounds on the convective heat transport is less than two times larger than the experimental results, the corresponding upper bound on the convective heat transport, obtained by Hunter and Riahi is about 10% higher than the bound obtained in this article. When Rayleigh number and Taylor number are high enough the upper bound on the convective heat transport ceases to depend on the boundary conditions.

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1 Introduction

The mathematical theory of the upper bounds on the convective heat transport was formulated by Howard [1] following the ideas of Malkus [2,3]. Howard obtained upper bounds on the convective heat transport through a horizontally infinite fluid layer on the basis of single-wavenumber solutions ($1-\alpha$-solutions) of a variational problem. Busse [4] introduced the more complicated multi-wavenumber (multi-α)-solutions of the variational problems of the optimum theory of turbulence. He has shown that the nonlinear Euler-Lagrange equations of the variational problems can have many coexisting solutions (with different numbers of wave numbers) for the same value of the Rayleigh number. Each of these solutions leads to an upper bound on the heat transport in a region of Rayleigh numbers. At the upper boundary of the nth region the upper bounds obtained by $n-\alpha$ and $n+1-\alpha$-solutions of the Euler-Lagrange ($n = 1, 2, 3...$) have the same value, and with increasing Rayleigh number the $n+1-\alpha$-solution leads to the upper bound on the convective heat transport. Chan [5] developed further the Howard-Busse method by an introduction of intermediate sublayers between the internal and boundary sublayers of the fields corresponding to a wave number of the multi-α-solutions of the Euler-Lagrange equations. The presence of rotation leads to complicated variational problems that allow solutions based on optimum fields without or with presence of Ekman sublayer in them [6–8]. The Howard-Busse method has been applied to many systems [9-25] and considerable efforts have been expanded to tighten the bounds by imposition of additional constraints onto the extremalizing vector fields. Busse [15] proposed to use as constraints the separate energy balances for toroidal and poloidal components of the velocity field. This idea is not applicable to all problems of the optimum theory of turbulence as has been shown by Kerswell and Soward [26] for the special case of plane Couette flow. Recently [27] it has been shown that for a horizontal layer of finite Prandtl number, heated from below, and rotating about a vertical axis, the use of the separate energy balances for the toroidal and poloidal components of the velocity field leads to an improving of the upper bounds on the convective heat transport. The realisation of the last idea opens a new possibility for tightening the upper bounds on the turbulent transport quantities.

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Another direction in the development of the optimum theory of turbulence was set by Doering and Constantin [28]. They developed a method based on the idea of a decomposition of the velocity into a steady background field that carries the inhomogeneous boundary conditions, and a homogeneous fluctuations field. The background field has to satisfy certain spectral constraints and a successful construction of an appropriate background field leads quickly to estimates of bounds of the turbulent quantities. This method as well as the energy balance parameter modification proposed by Nicodemus, Grossmann and Holthaus found many applications for shear flows and thermal convection problems [29-40]. Kerswell [41-43] discussed the relationship between the Doering-Constantin and Howard-Busse methods and formulated variational problems for Navier-Stokes equations. From the latest developments in this area we would like to mention the use of the Kolmogorov length scale as a constraint in the variational problem [37], the use of a smoothness constraint for lowering dissipation bounds for turbulent shear flows and bounds on the heat transport for the thermal convection [44,45], and the obtained upper bounds for: energy dissipation in a shear layer with suction [46]; mixing rate in turbulent stably stratified Couette flow [47]; bounds on the convective heat transport in containers [48] as well as the bounds for infinite Prandtl number convection with or without rotation [49,50]. The last bounds will be discussed in more details in the last section of this article.

In our previous article [8] we have presented an asymptotic theory for the $1-\alpha$-solution of the variational problem for the upper bounds on the convective heat transport in a heated from below horizontal fluid layer which rotates about the vertical axis with a constant angular velocity Ω (see Fig. 1) for the case of intermediate Taylor numbers. In this article we obtain upper bounds on the convective

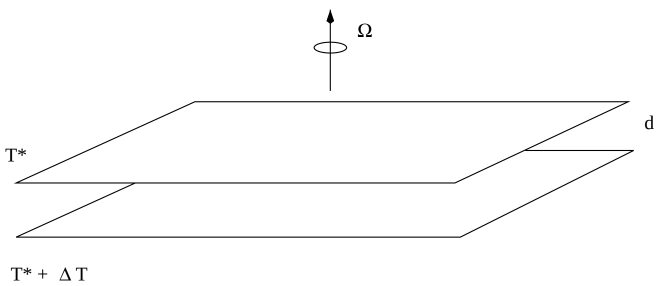

Fig. 1. The studied system. T^* is the temperature of the upper boundary of the fluid layer. Ω, d, and ΔT are the angular velocity, thickness of the fluid layer and the temperature difference between the bottom and top boundaries of the fluid layer.

heat transport for the interval of large Taylor numbers. The characteristic features of the optimum fields in the last region of Taylor numbers are as follows. We discuss below optimum fields that possess eight layers: four from the midplane of the fluid layer in the direction of the upper boundary of the fluid layer and four layers from the midplane of the fluid layer to the lower boundary of the fluid layer. The layers from the midplane of the fluid layer in the direction of one of the fluid boundaries are: internal layer, intermediate layer, Ekman layer and boundary layer Figure 2. We note that for the case of a fluid layer

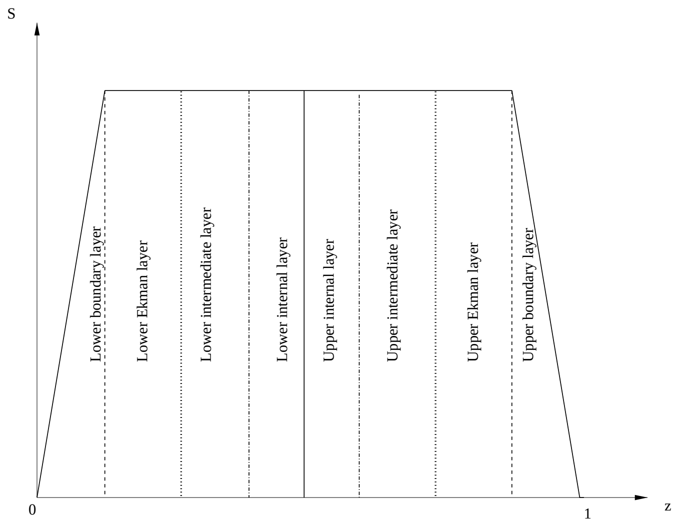

Fig. 2. A sketch of the optimum field $S = w_1\theta_1$ and its layer structure. In all layers except in the boundary ones $w_1\theta_1 \approx 1$.

with rigid lower boundary and stress-free upper boundary an asymmetry can exist in the sizes of the different layers of the studied optimum field. The reason for this are the different boundary conditions on the top and bottom boundaries of the fluid layer. When we discuss a fluid layer with two rigid boundaries or a fluid layer with two stress-free boundaries it is sufficient to consider only the half of the layers of the corresponding optimum field: these ones from the midplane of the fluid layer in the direction of the lower boundary of the fluid layer. We note further that the strong rotation can influence the internal layers of the optimum field. The intermediate layers expand in the direction of the corresponding internal layers. In most of the cases discussed below we treat the neighbouring intermediate and internal layers as one layer and we shall call this joint layer intermediate layer. The only exception will be for the case 3 of the discussion of the fluid layer with stress-free boundaries in Section 4. In this case we shall discuss optimum fields with and without an Ekman layer. For the case of an optimum field without an Ekman layer we shall treat the intermediate and internal layer as different ones. Another feature is that when the Rayleigh and Taylor numbers are high enough the boundary layer thickens, its width exceeds the width of the Ekman layer of the optimum field, and the upper bound on the convective heat transport decreases.

The structure of the article is as follows. In Section 2 we briefly formulate the variational problem. In Section 3 we discuss the basic case of our investigation: a fluid layer with rigid lower boundary and stress-free upper boundary. Section 4 is devoted to the case of a fluid layer with two stress-free boundaries and in Section 5 we discuss the fluid layer with two rigid boundaries. The intervals of validity of the obtained bounds are discussed in Section 6. Finally in Section 7 we discuss and compare the obtained results to other theoretical and experimental studies of the same or similar problem.

2 Mathematical formulation of the problem

Our investigation is based on the Boussinesq approximation to the equations of the fluid flow [51]. We denote the layer thickness as d, the thermometric conductivity and kinematic viscosity of the fluid as κ and ν, the acceleration of the gravity as g, the coefficient of thermal expansion as γ, the temperature difference between the upper and lower fluid boundary as ΔT and the mean density of the fluid as ρ. Taking d as an unit for length, κ/d as unit for velocity, d^2/κ as unit for time and $\rho\nu\kappa/d^2$ as unit for pressure we obtain the dimensionless form of the Boussinesq equations

$$\frac{1}{P}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\frac{\sqrt{Ta}}{2}\nabla p + \nabla^2\mathbf{u} + RT\mathbf{k} + \sqrt{Ta}\mathbf{u}\times\mathbf{k} \qquad \text{(1a)}$$

$$\frac{\partial \Theta}{\partial t} + \mathbf{u}\cdot\nabla\Theta = \nabla^2\Theta \qquad \text{(1b)}$$

$$\nabla\cdot\mathbf{u} = 0. \qquad \text{(1c)}$$

The boundary conditions can be rigid ones: $u_3 = \partial u_3/\partial z = T = 0$; stress-free ones: $u_3 = \partial^2 u_3/\partial z^2 = T = 0$; or mixed ones. $P = \nu/\kappa$ is the Prandtl number, $Ta = (4\Omega^2 d^4)/\nu^2$ is the Taylor number, $R = (\gamma g \Delta T d^3)/(\kappa\nu)$ is the Rayleigh number, p is the pressure, and $\mathbf{k}$ is the unit vector in the direction opposite to the gravity. Below we shall denote the averages of the quantities over the planes $z =$ const. as $\overline{q}$ and the averages over the fluid layer as $\langle q\rangle$. Θ is the total temperature field in (1a) and T is the deviation of the temperature field from its horizontal mean: $\Theta = \overline{\Theta} + T$. We formulate a variational problem for obtaining an upper bound on the convective heat transport, *i.e.* on the Nusselt number: $Nu = 1 + \langle u_3 T\rangle$. We assume that all necessary horizontal averages of the functions describing the flow exist, that the horizontal averages of the fluctuation quantities vanish, and that the flow is statistically steady in time and homogeneous in the horizontal averages.We introduce the above-mentioned decomposition of the total temperature field in the Boussinesq equations, multiply (1a) by the velocity $\mathbf{u}$ and the average over the fluid layer. Thus we obtain the following power integral

$$\langle|\nabla\mathbf{u}|^2\rangle = R\langle u_3 T\rangle.$$

Another power integral can be obtained by a multiplication of the heat equation by T and by averaging the result over the fluid layer. The obtained relationship contains the term $\langle u_3 T(\partial\overline{\Theta}/\partial z)\rangle$ that can be transformed by a horizontal averaging of the heat equation and integrating the obtained result with respect to z. As a result we obtain the second power integral

$$\langle|\nabla T|^2\rangle = \langle u_3 T\rangle^2 - \langle\overline{u_3 T}^2\rangle + \langle u_3 T\rangle.$$

In the case of infinite Prandtl number the Navier-Stokes equation becomes linear and we can include it as a constraint in the variational problem. We take into account the equation of continuity by the general representation of a solenoidal field $\mathbf{u}$ in terms of a poloidal and a toroidal component

$$\mathbf{u} = \nabla\times(\nabla\times\mathbf{k}\phi) + \nabla\times\mathbf{k}\psi.$$

We introduce the toroidal-poloidal decomposition of the velocity field into the linearised Navier-Stokes equation and perform the rescalings

$$\mathbf{u} = \langle u_3 T\rangle^{1/2} R^{1/2}\mathbf{v};\quad T = \langle u_3 T\rangle^{1/2} R^{-1/2}\theta.$$

Let us denote the z-component of the rescaled velocity field $\mathbf{v}$ as w. Taking the z-component of the horizontal curl and z-component of the double curl of the last result we obtain two relationships that help us to obtain the following expression for the rescaled second power integral

$$\langle u_3 T\rangle = \frac{\langle wT\rangle - (1/R)\langle|\nabla\theta|^2\rangle}{\langle(\langle w\theta\rangle - \overline{w\theta})^2\rangle}\,. \qquad (2)$$

We impose the condition: $\langle w\theta\rangle = 1$, denote the vertical component of the vorticity as $f = -\nabla_1^2\psi$ (∇_1^2 is the horizontal Laplacian) and write the variational problem as follows:

Find the maximum $F(R, Ta)$ of the variational functional

$$\mathcal{F}(w,\theta,f,R,Ta) = \frac{1-(1/R)\langle|\nabla\theta|^2\rangle}{\langle(1-\overline{w\theta}^2\rangle} + 2\lambda^*\langle w\theta - 1\rangle + 2\left\langle p^*\left(\nabla^2 f + \sqrt{Ta}\frac{\partial w}{\partial z}\right)\right\rangle + 2\left\langle q^*\left(\nabla^4 w + \nabla_1^2\theta - \sqrt{Ta}\frac{\partial f}{\partial z}\right)\right\rangle \qquad \text{(3a)}$$

among all fields w, θ, f subject to boundary conditions corresponding to rigid-stress-free, rigid or stress-free boundaries. For a fluid layer with rigid lower boundary and stress-free boundaries. For a fluid layer with rigid lower boundary and stress-free upper boundary

$$w_1 = \theta_1 = \frac{\mathrm{d}w_1}{\mathrm{d}z} = f_1 = 0, \qquad \text{(3b)}$$

at $z = 0$ and

$$w_1 = \theta_1 = \frac{\mathrm{d}^2 w_1}{\mathrm{d}z^2} = \frac{\mathrm{d}f_1}{\mathrm{d}z} = 0, \qquad \text{(3c)}$$

at $z = 1$. For a fluid layer with two stress-free boundaries the boundary conditions are (3c) at $z = 0, 1$. For a fluid layer with two rigid boundaries the boundary conditions are (3b) at $z = 0, 1$. The $1-\alpha$-solutions of the variational problem (for more details see [1,4,8]) have the form

$$w = w_1(z)\phi(x,y),\ \theta = \theta_1(z)\phi(x,y),\ f_1 = f_1(z)\phi(x,y)$$

where the average of the function ϕ^2 over the horizontal plane is equal to 1 and $\nabla_1^2\phi = -\alpha_1^2\phi$ (α_1 is the wave number connected to the $1-\alpha$-solution of the variational problem).

The Euler-Lagrange equations corresponding to the $1-\alpha$-solution of the variational problem are

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_1^2\right)^3 w_1 + Ta\frac{\mathrm{d}^2 w_1}{\mathrm{d}z^2} - \alpha_1^2\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_1^2\right)\theta_1 = 0, \tag{4a}$$

$$\frac{1}{RF_1}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_1^2\right)\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_1^2\right)^3 + Ta\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right]\theta_1 + \left[\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_1^2\right)^3 + Ta\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right]\left[w_1\left(1 - w_1\theta_1 + \frac{\lambda}{F}\right)\right] - \alpha_1^2\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_1^2\right)\left[\theta_1\left(1 - w_1\theta_1 + \frac{\lambda}{F_1}\right)\right] = 0, \tag{4b}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_1^2\right) f_1 + \sqrt{Ta}\frac{\mathrm{d}w_1}{\mathrm{d}z} = 0, \tag{4c}$$

where

$$\frac{1}{2} \leq \lambda = \frac{1}{2}\left(2 - \frac{1}{R}\langle \mid \nabla\theta \mid^2\rangle\right) \leq 1. \tag{4d}$$

F_1 is the maximum of the convective heat transport connected to the $1-\alpha$-solution of the variational problem.

The equations (4a–d) are the same for all three cases of boundary conditions, discussed in this article. For the case of a fluid layer with two stress-free boundaries these equations have been obtained by Chan [6]. We refer to [6] for more details about the process of obtaining of the Euler-Lagrange equations.

3 Fluid layer with rigid lower boundary and stress-free upper boundary

3.1 Case 1: Ekman layer is thicker than the boundary layer and $O(Ta^{1/8}) \ll \alpha_1 \ll O(R^{1/4})$

We can describe the internal and intermediate layers of the optimum fields with the same approximation to the Euler-Lagrange equations of the variational problem. In the internal and intermediate layers the assumption is $w_1\theta_1 = 1$. In addition we assume that in these layers the terms containing the derivatives and not containing the Taylor number are much smaller than the other terms in the Euler-Lagrange equations. The Euler-Lagrange equations (4a–c) become

$$-\alpha_1^6 w_1 + Ta\frac{\mathrm{d}^2 w_1}{\mathrm{d}z^2} + \alpha_1^4\theta_1 = 0, \tag{5a}$$

$$\alpha_1^2 f_1 = Ta^{1/2}\frac{\mathrm{d}w_1}{\mathrm{d}z}, \tag{5b}$$

$$w_1\theta_1 = 1. \tag{5c}$$

Let $w_1 = \breve{w}_1/\alpha_1$; $\theta_1 = \breve{\theta}_1\alpha_1$ and $\kappa^2 = Ta/\alpha_1^6$. From (5a) we obtain the equation

$$\kappa^2\frac{\mathrm{d}^2\breve{w}_1}{\mathrm{d}z^2} = \breve{w}_1 - \frac{1}{\breve{w}_1}\cdot \tag{6}$$

We note that $Ta \geq \alpha_1^6$ and κ is not a small quantity. The solution of (6) for small values of the coordinate z in the lower intermediate layer is

$$\breve{w}_1 \approx \frac{z+z_0}{\kappa}\sqrt{\ln\left[\frac{1}{(z+z_0)^2}\right] - \ln\ln\left[\frac{1}{(z+z_0)^2}\right]}. \tag{7a}$$

This solution satisfies also the first integral of (6) for small values of z. For the upper intermediate layer we introduce the coordinate $z^* = 1 - z$. The Euler-Lagrange equation (5a) can be reduced to an equation similar to (6). The solution of this equation for small values of z^*, which in addition satisfies also the first integral of (6), is

$$\breve{w}_1 \approx \frac{z^*+z_0^*}{\kappa}\sqrt{\ln\left[\frac{1}{(z^*+z_0^*)^2}\right] - \ln\ln\left[\frac{1}{(z^*+z_0^*)^2}\right]}. \tag{7b}$$

z_0 and z_0^* are constants which will be determined by the matching of the solutions of the Euler-Lagrange equations between the intermediate and Ekman layers of the optimum fields.

The Euler-Lagrange equations for the Ekman layers of the optimum fields are (5c) and

$$\frac{\mathrm{d}^4 w_1}{\mathrm{d}z^4} - Ta^{1/2}\frac{\mathrm{d}f_1}{\mathrm{d}z} = 0, \tag{8a}$$

$$\frac{\mathrm{d}^2 f1}{\mathrm{d}z^2} + Ta^{1/2}\frac{\mathrm{d}w_1}{\mathrm{d}z} = 0. \tag{8b}$$

For the lower Ekman layer we introduce the coordinate $\phi_l = Ta^{1/4}z/\sqrt{2}$ and obtain the following solutions of the Euler-Lagrange equations

$$w_1 = c_l\sqrt{2} - 2c_l\mathrm{e}^{-\phi_l}\cos(\phi_l - \pi/4), \tag{9a}$$

$$f_1 = Ta^{1/4}(2c_l - 2c_l\mathrm{e}^{-\phi_l}\cos(\phi_l)), \tag{9b}$$

which satisfy the boundary conditions $w_1 = f_1 = \mathrm{d}w_1/\mathrm{d}z = 0$ at $z = 0$. The constant c_l is determined by the matching of the solutions of the Euler-Lagrange equations between the intermediate and Ekman layers. This matching is performed for large ϕ_l and for small $z + z_0$. As a result we obtain the relationships

$$c_\mathrm{l} = \frac{z_0}{\alpha_1 \kappa \sqrt{2}} \sqrt{\ln\left(\frac{1}{z_0^2}\right) - \ln\ln\left(\frac{1}{z_0^2}\right)}, \tag{10a}$$

$$c_\mathrm{l} = \frac{Ta^{1/4}}{2\alpha_1^3 \kappa} \sqrt{\ln\left(\frac{1}{z_0^2}\right) - \ln\ln\left(\frac{1}{z_0^2}\right)}. \tag{10b}$$

(10a) and (10b) lead to

$$z_0 = \frac{Ta^{1/4}}{\sqrt{2}\alpha_1^2}, \tag{11a}$$

$$c_\mathrm{l} = \frac{1}{2Ta^{1/4}} \sqrt{\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right) - \ln\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right)}. \tag{11b}$$

For the upper Ekman layer the coordinate is $\phi_\mathrm{u} = Ta^{1/4}(1-z)/\sqrt{2}$ and the solutions of the Euler-Lagrange equation are

$$w_1 = c_\mathrm{u}(1 - \mathrm{e}^{-\phi_\mathrm{u}} \cos\phi_\mathrm{u}), \tag{12a}$$

$$f_1 = \frac{1}{2} Ta^{1/4} \Big[\sqrt{2}c_\mathrm{u}\mathrm{e}^{-\phi_\mathrm{u}}\cos\phi_\mathrm{u} - \sqrt{2}c_\mathrm{u}\mathrm{e}^{-\phi_\mathrm{u}}\sin\phi_\mathrm{u} + 2\sqrt{2}c_\mathrm{u}\phi_\mathrm{u} + 2kTa^{1/4}\Big] \tag{12b}$$

where k is a constant that can be estimated by the value of f at $\phi_\mathrm{u} = 0$. The only assumption that does not conflict to the other assumptions and constraints is that the terms $\sqrt{2}c_\mathrm{u}$ and $2kTa^{1/4}$ are of the same order and have different signs. Thus we obtain the estimation

$$k \propto -c_\mathrm{u}/(\sqrt{2}Ta^{1/4}).$$

The matching between the solutions for the upper Ekman and upper intermediate layers leads to the relationships

$$z_0^* = \frac{\sqrt{2}Ta^{1/4}}{\alpha_1^2}, \tag{13a}$$

$$c_\mathrm{u} = \frac{\sqrt{2}}{Ta^{1/4}} \sqrt{\ln\left(\frac{\alpha_1^4}{2Ta^{1/2}}\right) - \ln\ln\left(\frac{\alpha_1^4}{2Ta^{1/2}}\right)}. \tag{13b}$$

For the Euler-Lagrange equations in the boundary layers we assume that the terms which contain the highest derivatives are dominant in the considered interval of Taylor numbers. Thus we obtain

$$\frac{\mathrm{d}^4 w_1}{\mathrm{d}z^4} = 0, \tag{14a}$$

$$\frac{1}{RF_1}\frac{\mathrm{d}^2\theta_1}{\mathrm{d}z^2} + w_1(1 - w_1\theta_1) = 0, \tag{14b}$$

$$\frac{\mathrm{d}^2 f_1}{\mathrm{d}z^2} = 0. \tag{14c}$$

For the lower boundary layer we introduce the coordinate $\eta_\mathrm{u} = z/\delta_\mathrm{l}$ and the solution for w_1 is obtained by a Taylor expansions of the relationship for w_1 in the lower Ekman layer when $\phi_\mathrm{l} \to 0$, taking the leading order of this expansion, and changing the coordinate from ϕ_l to η_l. The result is

$$w_1 = \frac{1}{\sqrt{2}} c_\mathrm{l} Ta^{1/2} \delta_\mathrm{l}^2 \eta_\mathrm{l}^2. \tag{15a}$$

(15a) satisfies the corresponding Euler-Lagrange equation as well as the rigid boundary conditions on w_1 when $\eta_\mathrm{l} = 0$. The expression for f_1 is

$$f_1 = \sqrt{2} c_\mathrm{l} Ta^{1/2} \delta_\mathrm{l} \eta_\mathrm{l}. \tag{15b}$$

For the upper boundary layer we introduce the coordinate $\eta_\mathrm{u} = (1-z)/\delta_\mathrm{u}$ and from the relationships for w_1 and f_1 in the Ekman layer the following relationships for the upper boundary layer are obtained

$$w_1 = \frac{c_\mathrm{u}\delta_\mathrm{u} Ta^{1/4}\eta_\mathrm{u}}{\sqrt{2}}, \tag{16a}$$

$$f_1 = \frac{1}{2} Ta^{1/4} \left[2kTa^{1/4} + \sqrt{2}c_\mathrm{u}\left(1 + \frac{Ta^{1/2}\delta_\mathrm{u}^2\eta_\mathrm{u}^2}{2}\right)\right]. \tag{16b}$$

(16b) is obtained from (12b) for small values of ϕ_u and representing $\mathrm{e}^{-\phi_\mathrm{u}}$, $\sin(\phi_\mathrm{u})$ and $\cos(\phi_\mathrm{u})$ by means of Taylor series. In order to solve the equation (14b) for the lower boundary layer we introduce the relationship

$$RF_1 \delta_\mathrm{l}^6 c_\mathrm{l}^2 Ta = 2, \tag{17a}$$

and thus we have to solve the equation

$$\frac{\mathrm{d}^2\hat{\theta}_1}{\mathrm{d}\eta_\mathrm{l}^2} + \eta_\mathrm{l}^2(1 - \eta_\mathrm{l}^2\hat{\theta}_1) = 0, \tag{17b}$$

where $\theta_1 = \sqrt{2}\hat{\theta}_1/(c_l \delta_l^2 Ta^{1/2})$ and with corresponding boundary condition at $\eta_l = 0$. In the same manner for the upper boundary layer

$$RF_1 c_u^2 \delta_u^4 Ta^{1/2} = 2. \tag{17c}$$

The resulting equation is

$$\frac{d^2\hat{\theta}_1}{d\eta_u^2} + \eta_u(1 - \eta_u\hat{\theta}_1) = 0, \tag{17d}$$

with corresponding boundary conditions and

$$\theta_1 = \sqrt{2}\hat{\theta}_1/(c_u \delta_u Ta^{1/4}).$$

In order to obtain relationships for the upper bound on the convective heat transport and for corresponding wave number and boundary layer thicknesses we have to calculate the integrals in the expression for F

$$F = \frac{1 - (1/R)\langle | \nabla\theta_1 |^2\rangle}{\langle (1 - \overline{w_1\theta_1})^2\rangle} \cdot \tag{18a}$$

F can be written as

$$F = \frac{1 - \alpha_1^4/R}{\delta_l D_l + \delta_u D_u}, \tag{18b}$$

where

$$D_l = I_l + J_l, \quad D_u + I_u + J_u, \tag{18c}$$

$$I_l = \int_0^\infty d\eta_l (1 - \eta_l^2\hat{\theta}_1)^2 = 0.9255, \tag{18d}$$

$$I_u = \int_0^\infty d\eta_u (1 - \eta_u\hat{\theta}_1)^2 = 0.79635, \tag{18e}$$

$$J_l = \int_0^\infty d\eta_l \left(\frac{d\hat{\theta}_1}{d\eta_l^2}\right)^2 = 0.1851, \tag{18f}$$

$$J_u = \int_0^\infty d\eta_u \left(\frac{d\hat{\theta}_1}{d\eta_u^2}\right)^2 = 0.2635. \tag{18g}$$

From (17a, 17c) and (18b) we determine F, δ_u and δ_l. Assuming that the thicknesses of the upper and lower boundary layers are of the same order, we obtain

$$\delta_l = 2^{3/5}(D_u + D_l)^{1/5}(R - \alpha_1^4)^{-1/5} Ta^{-1/10} \times \left[\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right) - \ln\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right)\right]^{-1/5}, \tag{19a}$$

$$\delta_u = 2^{3/20}(D_u + D_l)^{3/10}(R - \alpha_1^4)^{-3/10} Ta^{-1/40} \times \left[\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right) - \ln\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right)\right]^{-1/20} \times \left[\ln\left(\frac{\alpha_1^4}{2Ta^{1/2}}\right) - \ln\ln\left(\frac{\alpha_1^4}{2Ta^{1/2}}\right)\right]^{-1/4}, \tag{19b}$$

$$F = 2^{-3/5}(D_u + D_l)^{-6/5} R^{-1}(R - \alpha_1^4)^{6/5} Ta^{1/10} \times \left[\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right) - \ln\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right)\right]^{1/5}. \tag{19c}$$

In order to obtain a maximum value of the convective heat transport we note that F_1 increases with increasing α_1 and that we must have $\alpha_1^4 \le O(R)$. Thus when $\alpha_1 \propto O(Ta^{1/6})$ in the interval

$$O(Ta^{1/8}) \ll \alpha_1 \le O(R^{1/4}). \tag{20a}$$

we obtain

$$\delta_u \propto 2^{3/20}(D_u + D_l)^{3/10} R^{-3/10} Ta^{-1/40}\Big[\ln(2Ta^{1/6}) - \ln\ln(2Ta^{1/6})\Big]^{-1/20}\Big[\ln(Ta^{1/6}/2) - \ln\ln(Ta^{1/6}/2)\Big]^{-1/4}, \tag{20b}$$

$$\delta_l \propto 2^{3/5}(D_u + D_l)^{1/5} R^{-1/5} Ta^{-1/10}\Big[\ln(2Ta^{1/6}) - \ln\ln(2Ta^{1/6})\Big]^{-1/5}, \tag{20c}$$

$$F_1 \propto 2^{-3/5}(D_u + D_l)^{-6/5} R^{1/5} Ta^{1/10}\Big[\ln(2Ta^{1/6}) - \ln\ln(2Ta^{1/6})\Big]^{1/5}. \tag{20d}$$

3.2 Case 2: Ekman layer is thicker than the boundary layer and $O((Ta/R)^{1/2}) \ll \alpha_1 \ll O(Ta^{1/8})$

In the intermediate and the Ekman layers we have $w_1\theta_1 \approx 1$. For the lower intermediate layer $w_1 = w_{0l}/\alpha_1$; $\theta_1 = \alpha_1/w_{0l}$ where w_{0l} is a small constant. In addition the following condition must be satisfied: $\alpha_1^2 f_1 \ll 1$. The Euler-Lagrange equation for f_1 is

$$\left(\frac{d^2}{dz^2} - \alpha_1^2\right)^2 w_1 - \alpha_1^2\theta_1 - Ta\frac{df_1}{dz} = 0, \tag{21a}$$

and the solution for f_1 is

$$f_1 = \frac{\alpha_1^3}{w_{0l} Ta}\left(\frac{1}{2} - z\right). \quad (21b)$$

For the upper intermediate layer we have $w_1 = w_{0u}/\alpha_1$; $\theta_1 = \alpha_1/w_{0u}$ where w_{0u} is a small constant. The solution of (21a) for this layer is

$$f_1 = \frac{\alpha_1^3}{w_{0u} Ta}\left(z - \frac{1}{2}\right). \quad (21c)$$

The coordinate for the lower Ekman layer is $\phi_l = Ta^{1/4} z/\sqrt{2}$ and the solution of the Euler-Lagrange equations are (9a) and (9b). The matching of the solutions between the intermediate and Ekman layers leads to the relationships

$$w_{0l} = 2^{-3/4}\alpha_1^2 Ta^{-3/8}, \quad (22a)$$

$$c_l = 2^{-5/4}\alpha_1 Ta^{-3/8}. \quad (22b)$$

For the upper Ekman layer the coordinate is $\phi_u = Ta^{1/4}(1-z)/\sqrt{2}$ and the solutions of the Euler-Lagrange equations are (12a) and (12b). The matching of the solutions between the upper intermediate and Ekman layers leads us to the relationships

$$w_{0u} = 2^{-1/4}\alpha_1^2 Ta^{-5/8}, \quad (23a)$$

$$c_u = 2^{-1/4}\alpha_1 Ta^{-5/8}. \quad (23b)$$

For the upper and lower boundary layers we introduce the coordinates $\eta_{u,l} = z/\delta_{u,l}$ and match the solutions of the Euler-Lagrange equations for the Ekman layers in the boundary ones. Thus for the lower boundary layer we obtain (15a) and (15b) for w_1 and f_1 as well as the equation (17b) for $\hat{\theta}_1$ and the relationship (17a). The corresponding relationships and equations for the upper boundary layer are (16a, 16b, 17c) and (17b). The relationships for the convective heat transport and for the thicknesses of the boundary layers can be obtained by means of (17a, 17c) and (18b). The results are

$$\delta_l = 2^{7/10}(D_u + D_l)^{1/5}\alpha_1^{-2/5} Ta^{-1/20}(R - \alpha_1^4)^{-1/5}, \quad (24a)$$

$$\delta_u = 2^{11/20}(D_u + D_l)^{3/10}\alpha_1^{-3/5} Ta^{7/40}(R - \alpha_1^4)^{-3/10}, \quad (24b)$$

$$F_1 = 2^{-7/10}(D_u + D_l)^{-6/5}\alpha_1^{2/5} Ta^{1/20} R^{1/5}(1 - \alpha_1^4/R)^{6/5}. \quad (24c)$$

As F_1 increases with increasing α_1 a maximum of F_1 can be obtained when $\alpha_1 \to O(Ta^{1/8})$. Then

$$F_1 \propto 2^{-7/10}(D_u + D_l)^{-6/5} Ta^{1/10} R^{1/5}, \quad (25a)$$

$$\delta_l \propto 2^{7/10}(D_u + D_l)^{1/5} Ta^{-1/10} R^{-1/5}, \quad (25b)$$

$$\delta_u \propto 2^{11/20}(D_u + D_l)^{3/10} Ta^{1/10} R^{-3/10}. \quad (25c)$$

3.3 Case 3: Boundary layer is thicker than the Ekman layer and $O((R \ln R)^{4/3}) \ll O(Ta) \ll O(R^{3/2})$

In the intermediate and Ekman layers $w_1\theta_1 \approx 1$. The matching of the solutions of the Euler-Lagrange equations of the variational problem is performed between intermediate and boundary layers. The Euler-Lagrange equations for the intermediate layers of the optimum fields are obtained by the assumption that the terms containing derivatives and not containing the Taylor number are much smaller than the other terms. Thus the equations are (5a–c) and the solution for w_1 in the lower intermediate layer is

$$w_1 = \frac{\xi_l}{\alpha_1}\sqrt{\ln\left(\frac{1}{\xi_l^2}\right) - \ln\ln\left(\frac{1}{\xi_l^2}\right)}. \quad (26)$$

$\xi_l = \alpha_1^3 z/Ta^{1/2}$ is the coordinate in the lower intermediate layer of the optimum field $w_1\theta_1$. The solutions of the Euler-Lagrange equations for w_1 and f_1 in the lower Ekman layer are (9a) and (9b). Neglecting the terms which contain derivatives and do not contain the Taylor number we obtain the simplified form of the Euler-Lagrange equations for the boundary of the optimum field

$$-\alpha_1^6 w_1 + Ta\frac{d^2 w_1}{dz^2} + \alpha_1^4\theta_1 = 0, \quad (27a)$$

$$-\alpha_1^2 f_1 + Ta^{1/2}\frac{dw_1}{dz} = 0, \quad (27b)$$

$$-\frac{\alpha_1^2}{RF_1}\frac{d^2\theta_1}{dz^2} + \frac{d^2}{dz^2}\left[w_1(1 - w_1\theta_1)\right] = 0. \quad (27c)$$

For the lower boundary layer we introduce the coordinate $\eta_l = z/\delta_l$. δ_l depends on the Rayleigh and Taylor numbers and can be connected to the thickness of the lower boundary layer of the optimum field. We rescale w_1 and θ_1: $w_1 = A_l\hat{w}_1$; $\theta_1 = \hat{\theta}_1/A_l$. A_l depends on the wave numbers α_1, Taylor number, and δ_l

$$A_l = \frac{\alpha_1^2\delta_l}{Ta^{1/2}}\sqrt{\ln\left(\frac{Ta}{\alpha_1^6\delta_l^2}\right) - \ln\ln\left(\frac{Ta}{\alpha_1^6\delta_l^2}\right)}. \quad (28a)$$

A_l must satisfy the relationship

$$\alpha_1^2 = RFA_\mathrm{l}^2. \tag{28b}$$

The solutions for $\hat{w}_1$ and $\hat{\theta}_1$ are: $\hat{w}_1 = \eta_\mathrm{l}$; $\hat{\theta}_1 = \eta_\mathrm{l}/(1+\eta_\mathrm{l}^2)$.

For the upper intermediate layer the coordinate is $\xi_\mathrm{u} = \alpha_1^3(1-z)/Ta^{1/2}$ and the solution for w_1 is

$$w_1 = \frac{\xi_\mathrm{u}}{\alpha_1}\sqrt{\ln\left(\frac{1}{\xi_\mathrm{u}^2}\right) - \ln\ln\left(\frac{1}{\xi_\mathrm{u}^2}\right)}. \tag{29}$$

The solutions for the upper Ekman layers are (12a) and (12b). For the upper boundary layer: $w_1 = A_\mathrm{u}\hat{w}_1$; $\theta_1 = \hat{\theta}_1/A_\mathrm{u}$ and

$$A_\mathrm{u} = \frac{\alpha_1^2\delta_\mathrm{u}}{Ta^{1/2}}\sqrt{\ln\left(\frac{Ta}{\alpha_1^6\delta_\mathrm{u}^2}\right) - \ln\ln\left(\frac{Ta}{\alpha_1^6\delta_\mathrm{u}^2}\right)}. \tag{30a}$$

In addition the following relationship must be satisfied

$$\alpha_1^2 = RFA_\mathrm{u}^2. \tag{30b}$$

The solutions for $\hat{w}_1$ and $\hat{\theta}_1$ are: $\hat{w}_1 = \eta_\mathrm{u}$; $\hat{\theta}_1 = \eta_\mathrm{u}/(1+\eta_\mathrm{u}^2)$. The solution for $\hat{w}_1$ holds in the whole boundary layer except for a small region near the fluid boundary where the stress-free boundary conditions must be satisfied.

The relationships for the convective heat transport F_1 and for the thicknesses of the boundary layers of the optimum field $\delta_{\mathrm{u,l}}$ can be obtained by (28a, 30a) and

$$F_1 = \frac{4}{\pi}\frac{1-\alpha_1^4/R}{\delta_\mathrm{u} + \delta_\mathrm{l} + \frac{1}{2\delta_\mathrm{l}\alpha_1^2} + \frac{1}{2\delta_\mathrm{u}\alpha_1^2}}, \tag{31a}$$

$$\frac{\partial F_1}{\partial \alpha_1} = 0. \tag{31b}$$

Assuming that the thicknesses of the boundary layers are of the same order we obtain the solutions

$$\alpha_1 = \left(\frac{R}{5}\right)^{1/4}, \tag{32a}$$

$$F = \frac{64R^{3/2}}{25\sqrt{5}\pi^2 Ta} \times \left[\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right) - \ln\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)\right], \tag{32b}$$

and for the thicknesses of the boundary layers

$$\delta_\mathrm{u} \propto \delta_\mathrm{l} = \frac{5\sqrt{5}\pi Ta}{8R^{3/2}} \times \left[\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right) - \ln\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)\right]. \tag{32c}$$

3.4 Case 4: Boundary layer is thicker than the Ekman layer and $O(R^{1/4}) \gg O(Ta^{1/6}) \gg \alpha_1 \gg O((Ta/R)^{1/2}) \gg O(R^{1/8})$

For the layers of the optimum fields from the midplane of the fluid layer in the direction of the rigid boundary of the fluid layer we obtain the following solution of the Euler-Lagrange equation of the variational problem for w_1 in the intermediate layer

$$w_1 = \frac{z+z_0}{\kappa\alpha_1}\sqrt{\ln\left(\frac{1}{(z+z_0)^2}\right) - \ln\ln\left(\frac{1}{(z+z_0)^2}\right)}. \tag{33a}$$

We note that $\kappa^2 = Ta/\alpha_1^6$ is a large quantity and (33a) is a solution of the corresponding Euler-Lagrange equations for small values of z. This solution satisfies also the first integral of the corresponding Euler-Lagrange equation. The matching between the intermediate and boundary layer leads to the following relationship for w_1

$$w_1 = \frac{\alpha_1^2\delta_\mathrm{l}\eta_\mathrm{l}}{Ta^{1/2}}\sqrt{\ln\left(\frac{1}{\delta_\mathrm{l}^2}\right) - \ln\ln\left(\frac{1}{\delta_\mathrm{l}^2}\right)}. \tag{33b}$$

In addition $\alpha_1^2/(RF_1A_\mathrm{l}^2) = 1$. $\eta_\mathrm{l} = z/\delta_\mathrm{l}$ is the coordinate in the lower boundary layer of the optimum field. For the layers from the midplane of the fluid layer in the direction of the upper stress-free boundary of the fluid layer we have as a solution for w_1 in the corresponding intermediate layer

$$w_1 = \frac{z+z_0^*}{\kappa\alpha_1}\sqrt{\ln\left(\frac{1}{(z+z_0^*)^2}\right) - \ln\ln\left(\frac{1}{(z+z_0^*)^2}\right)}. \tag{34a}$$

The matching between the upper intermediate and boundary layers leads to the solution for w_1 in the upper boundary layer

$$w_1 = \frac{\alpha_1^2\delta_\mathrm{u}\eta_\mathrm{u}}{Ta^{1/2}}\sqrt{\ln\left(\frac{1}{\delta_\mathrm{u}^2}\right) - \ln\ln\left(\frac{1}{\delta_\mathrm{u}^2}\right)}. \tag{34b}$$

In addition $\alpha_1^2/(RF_1A_\mathrm{u}^2) = 1$ and the coordinate in this layer is $\eta_\mathrm{u} = (1-z)/\delta_\mathrm{u}$. The equations for the calculation of the upper bounds are

$$F_1 = \frac{2(1-\alpha_1^4/R)}{\pi(\delta_\mathrm{l} + 1/(2\delta_\mathrm{l}\alpha_1^2))}, \tag{35a}$$

$$\frac{\alpha_1^2}{RF_1A_\mathrm{u}^2} = \frac{\alpha_1^2}{RFA_\mathrm{l}^2} = 1, \tag{35b}$$

$$A_\mathrm{l} = \frac{\alpha_1^2\delta_\mathrm{l}}{Ta^{1/2}}\sqrt{\ln\left(\frac{1}{\delta_\mathrm{l}^2}\right) - \ln\ln\left(\frac{1}{\delta_\mathrm{l}^2}\right)}, \tag{35c}$$

$$A_{\mathrm{u}} = \frac{\alpha_1^2 \delta_{\mathrm{u}}}{Ta^{1/2}} \sqrt{\ln\left(\frac{1}{\delta_{\mathrm{u}}^2}\right) - \ln\ln\left(\frac{1}{\delta_{\mathrm{u}}^2}\right)}. \qquad (35d)$$

The obtained solutions are

$$F = \frac{4R\alpha_1^2}{\pi^2 Ta}\left[\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right) - \ln\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right)\right], \qquad (36a)$$

$$\delta_{\mathrm{u}} \propto \delta_{\mathrm{l}} = \frac{\pi Ta}{2R\alpha_1^2}\left[\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right) - \ln\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right)\right]^{-1}. \qquad (36b)$$

4 Fluid layer with two stress-free boundaries

4.1 Case 1: Ekman layer is thicker than the boundary layer and $O(Ta^{1/8}) \ll \alpha_1 \ll O(R^{1/4})$

The rotation effects are important in the internal layers of the optimum field and the intermediate layers expand in the direction of the internal layers. We assume that $w_1\theta_1 \approx 1$ holds in this region of the fluid layer. Because of the symmetry it is sufficient to discuss the layers of the optimum fields from the midplane of the fluid layer to the lower rigid boundary of the fluid layer. Assuming that the terms in the Euler-Lagrange equations which contain derivatives and do not contain the Taylor number are much smaller than the other terms we obtain the equations (5a–c). They can be reduced to the equation (6) possessing solution (7a). Assuming that $w_1\theta_1 \approx 1$ holds also in the Ekman layers we obtain the equations (8a, 8b) plus corresponding boundary conditions. The solutions of the Euler-Lagrange equations are

$$w_1 = c[1 - \mathrm{e}^{-\phi}\cos(\phi)], \qquad (37a)$$

$$f_1 = \frac{1}{2}Ta^{1/4}\Big[-\sqrt{2}c\mathrm{e}^{-\phi}\cos(\phi) + \sqrt{2}c\mathrm{e}^{-\phi}\sin(\phi) - 2\sqrt{2}c\phi + 2kTa^{1/4}\Big]. \qquad (37b)$$

$\phi = Ta^{1/4}z/\sqrt{2}$ is the coordinate in the Ekman layer and k is an integration constant determined by the boundary condition for f_1 on $\phi = 0$. The constants c and z_0 are determined by a matching of the solutions of the Euler-Lagrange equations between the intermediate and Ekman layers of the optimum field. The result is

$$z_0 = \frac{\sqrt{2}Ta^{1/4}}{\alpha_1^2}, \qquad (38a)$$

$$c = \frac{\sqrt{2}}{Ta^{1/4}}\sqrt{\ln\left(\frac{\alpha_1^4}{2Ta^{1/2}}\right) - \ln\ln\left(\frac{\alpha_1^4}{2Ta^{1/2}}\right)}. \qquad (38b)$$

The assumption in the boundary layer is that the significant terms in the Euler-Lagrange equations are the terms containing the highest derivatives. Thus the Euler-Lagrange equations are reduced to (14a–c). Introducing the boundary-layer coordinate $\eta = z/\delta$ where δ can be connected to the thickness of the boundary layer of the field $w_1\theta_1$ and matching the solutions of the Euler-Lagrange equations of the variational problem between the Ekman and boundary layers we obtain the following relationships for w_1 and f_1 in the boundary layer

$$w_1 = \frac{c\delta Ta^{1/4}\eta}{\sqrt{2}}, \qquad (39a)$$

$$f_1 = Ta^{1/2}\left[k - \frac{c}{\sqrt{2}Ta^{1/4}}\left(1 + \frac{\delta^2 Ta^{1/2}\eta^2}{2}\right)\right]. \qquad (39b)$$

Assuming that (41b) is valid we obtain the following equation for

$$\theta_1 = \sqrt{2}\hat{\theta}_1/c\delta Ta^{1/4}$$

$$\frac{\mathrm{d}^2\hat{\theta}_1}{\mathrm{d}\eta^2} + \eta(1 - \eta\hat{\theta}_1) = 0 \qquad (40)$$

with the boundary condition $\hat{\theta}_1 = 0$ at $\eta = 0$.

The equations we have to solve for the boundary layer thickness and for the convective heat transport are

$$F_1 = \frac{1 - \alpha_1^4/R}{2\delta D}, \qquad (41a)$$

where $D = I + J$ and I and J are the same as I_{u} and J_{u} from (18e) and (18g).

$$c^2\delta^4 Ta^{1/2} R F_1 = 2. \qquad (41b)$$

The solutions are

$$\delta = (2D)^{1/3}R^{-1/3}(1 - \alpha_1^4/R)^{-1/3} \times \left[\ln\left(\frac{\alpha_1^4}{2Ta^{1/2}}\right) - \ln\ln\left(\frac{\alpha_1^4}{2Ta^{1/2}}\right)\right]^{-1/3}, \qquad (42a)$$

$$F_1 = (2D)^{-4/3}R^{1/3}(1 - \alpha_1^4/R)^{4/3} \times \left[\ln\left(\frac{\alpha_1^4}{2Ta^{1/2}}\right) - \ln\ln\left(\frac{\alpha_1^4}{2Ta^{1/2}}\right)\right]^{1/3}. \qquad (42b)$$

In order to obtain a maximum for the convective heat transport we take into account that F_1 increases with increasing α_1 and that the relationship $\alpha_1 \leq O(R^{1/4})$ must be satisfied. Thus in the discussed in this section interval for α and when $\alpha_1 \propto O(Ta^{1/6})$ we obtain

$$\delta \propto (2D)^{1/3} R^{-1/3} \times \left[\ln(Ta^{1/6}/2) - \ln\ln(Ta^{1/6}/2)\right]^{-1/3}, \quad (43a)$$

$$F_1 \propto (2D)^{-4/3} R^{1/3} \times \left[\ln(Ta^{1/6}/2) - \ln\ln(Ta^{1/6}/2)\right]^{1/3}. \quad (43b)$$

4.2 Case 2: Ekman layer is thicker than the boundary layer and $O((Ta/R)^{1/2}) \ll \alpha_1 \ll O(Ta^{1/8})$

In the intermediate layer of the optimum field $w_1\theta_1 \approx 1$ and the solutions of the Euler-Lagrange equations of the variational problem are

$$w_1 = w_0/\alpha_1;\ \theta_1 = \alpha_1/w_0;\ f_1 = \alpha_1^3(1/2 - z)/(w_0 Ta).$$

Solving the Euler-Lagrange equations for the Ekman layer and matching the solutions on the boundary between the intermediate and Ekman layers we obtain

$$w_0 = 2^{-1/4}\alpha_1^2 Ta^{-5/8}.$$

This relationship as well as the obtained by the solving of the Euler-Lagrange equations

$$c^2\delta^4 Ta^{1/2} R F_1 = 2$$

with $c = 2^{-1/4}\alpha_1 Ta^{-5/8}$, and finally

$$F = (1 - \alpha_1^4/R)/(2\delta D),$$

with $D = I + J$, and I and J are the same as I_u and J_u from (18e) and (18g), lead to the following results for the thickness of the boundary layer of the optimum field and for the upper bound on the convective heat transport

$$\delta = 2^{5/6} D^{1/3} (R - \alpha_1^4)^{-1/3} Ta^{1/4} \alpha_1^{-2/3}, \quad (44a)$$

$$F = 2^{-11/6} D^{-4/3} \alpha_1^{2/3} R^{1/3} (1 - \alpha_1^4/R)^{4/3} Ta^{-1/4}. \quad (44b)$$

As the convective heat transport F_1 increases with increasing α_1 a maximum of F_1 can be obtained when $\alpha_1 \to O(Ta^{1/8})$. Then

$$F \propto 2^{-11/6} D^{-4/3} R^{1/3} Ta^{-1/6}, \quad (45a)$$

$$\delta = 2^{5/6} D^{1/3} R^{-1/3} Ta^{1/6}. \quad (45b)$$

4.3 Case 3: Boundary layer is thicker than the Ekman layer and $O((R \ln R)^{4/3}) \ll Ta \ll O(R^{3/2})$

Here we have two possibilities: optimum fields without Ekman layer (three layer optimum fields) and optimum fields with Ekman layer (four layers optimum fields). Let us consider first the three-layers optimum fields. The layers of the optimum fields are: internal, intermediate and boundary ones. For the internal layer we assume $w_1\theta_1 \approx 1$, $\lambda \ll F$ and that the term containing derivatives are vanishingly small in comparison to the other terms in the Euler-Lagrange equations. Thus we obtain the solutions for this layer

$$w_1 = 1/\alpha_1;\ \theta_1 = \alpha_1;\ f_1 = 0.$$

For the intermediate layers the coordinate is $\xi = z\alpha_1^3/Ta^{1/2}$. Remembering that $Ta \gg \alpha_1^6$ and solving the obtained equation for w_1 we obtain for the case of small ξ

$$w_1 = \frac{\xi}{\alpha_1}\sqrt{\ln\left(\frac{1}{\xi^2}\right) - \ln\ln\left(\frac{1}{\xi^2}\right)}. \quad (46)$$

For the boundary layers the coordinate is $\eta = Ta^{1/2} z/(\alpha_1\delta)$. Let $w_1 = A\hat{w}_1$ and $\theta_1 = \hat{\theta}_1/A$ where A is some function to be determined by a matching of the solutions of the Euler-Lagrange equations between intermediate and the boundary layers. Assuming that the terms containing the highest derivatives are dominant we obtain for the Euler-Lagrange equations for the boundary layers

$$\frac{d^2\hat{w}_1}{d\eta^2} = 0, \quad (47a)$$

$$\frac{\alpha_1^2}{RFA^2}\frac{d^2\hat{\theta}_1}{d\eta^2} - \frac{d^2}{d\eta^2}[\hat{w}_1(1 - \hat{w}_1\hat{\theta}_1)] = 0, \quad (47b)$$

The solution for $\hat{w}_1$ is $\hat{w}_1 = \eta$. Matching the solutions for w_1 between intermediate and boundary layers we obtain for A

$$A = \frac{\alpha_1^3\delta}{Ta}\sqrt{\ln\left(\frac{Ta^2}{\alpha_1^8\delta^2}\right) - \ln\ln\left(\frac{Ta^2}{\alpha_1^8\delta^2}\right)}. \quad (48a)$$

Thus for $\hat{\theta}_1$ we obtain from (47b)

$$\frac{Ta^2}{RF\delta^2\alpha_1^4}\left[\ln\left(\frac{Ta^2}{\alpha_1^8\delta^2}\right) - \ln\ln\left(\frac{Ta^2}{\alpha_1^8\delta^2}\right)\right]^{-1}\hat{\theta}_1 = \hat{w}_1(1 - \hat{w}_1\hat{\theta}_1). \quad (48b)$$

Assuming that

$$\frac{Ta^2}{RF\delta^2\alpha_1^4}\left[\ln\left(\frac{Ta^2}{\alpha_1^8\delta^2}\right) - \ln\ln\left(\frac{Ta^2}{\alpha_1^8\delta^2}\right)\right]^{-1} = 1, \quad (49a)$$

we obtain the solution for $\hat{\theta}_1$

$$\hat{\theta}_1 = \frac{\eta}{1+\eta^2}, \tag{49b}$$

which satisfies the corresponding boundary conditions. The equations for the upper bounds on the convective heat transport and for the thickness of the boundary layer of the optimum field $w_1\theta_1$ are (49a) and

$$F_1 = \frac{2Ta^{1/2}}{\pi\alpha_1\delta R}\left[1-\left(\alpha_1^4+\frac{\pi Ta^{1/2}}{4\alpha_1\delta A^2}\right)\right] \tag{50a}$$

$$\frac{\partial F_1}{\partial\alpha_1} = 0. \tag{50b}$$

The solutions for the upper bound on the convective heat transport, for the thickness of the boundary layer, and for the optimum wave number are

$$\alpha_1 = \left(\frac{R}{5}\right)^{1/4}, \tag{51a}$$

$$F = \frac{64R^{3/2}}{25\sqrt{5}\pi^2 Ta} \times\left[\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)-\ln\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)\right], \tag{51b}$$

and for the thicknesses of the boundary layers which are assumed to be of the same order

$$\delta = \frac{5\sqrt{5}\pi Ta}{8R^{3/2}} \times\left[\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)-\ln\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)\right]. \tag{51c}$$

For the case of four-layers optimum field the internal layer is included in the intermediate layer as an effect of the strong rotation. In the intermediate layer the solution for w_1 is

$$w_1 = \frac{\xi}{\alpha_1}\sqrt{\ln\left(\frac{1}{\xi^2}\right)-\ln\ln\left(\frac{1}{\xi^2}\right)} \tag{52}$$

where $\xi = \alpha_1^3 z/Ta^{1/2}$ is the coordinate in the intermediate layer of the optimum field. For the Ekman layer the coordinate is $\phi = Ta^{1/4}z/\sqrt{2}$ and the solutions of the Euler-Lagrange equations are

$$w_1 = c\left[1-\mathrm{e}^{\phi}\cos(\phi)\right] \tag{53a}$$

$$f = \frac{Ta^{1/4}}{2}\Big[-\sqrt{2}c\mathrm{e}^{-\phi}\cos(\phi) +\sqrt{2}c\mathrm{e}^{-\phi}\sin(\phi)-2\sqrt{2}c\phi+2kTa^{1/4}\Big] \tag{53b}$$

where

$$k = \frac{c}{\sqrt{2}Ta^{1/4}} \tag{53c}$$

and

$$c = \frac{\sqrt{2}}{Ta^{1/4}}\sqrt{\ln\left(\frac{Ta^{1/2}}{2\alpha_1^2}\right)-\ln\ln\left(\frac{Ta^{1/2}}{2\alpha_1^2}\right)}. \tag{53d}$$

The solutions of the Euler-Lagrange equations for the boundary layer of the optimum field are $w_1 = A\eta$; $\theta_1 = \hat{\eta}/(A(1+\eta^2))$ where $\eta = z/\delta$ is the coordinate in the boundary layer of the optimum field and

$$A = \frac{\alpha_1^2\delta}{Ta^{1/2}}\sqrt{\ln\left(\frac{Ta}{\alpha_1^6\delta^2}\right)-\ln\ln\left(\frac{Ta}{\alpha_1^6\delta^2}\right)}. \tag{54}$$

For calculation of α_1 and F_1 we have the equations

$$\frac{\alpha_1^2}{RF_1A^2} = 1, \tag{55a}$$

$$F_1 = \frac{1-(1/R)\langle\mid\nabla\theta_1\mid^2\rangle}{\langle(1-w_1\theta_1)^2\rangle}, \tag{55b}$$

$$\frac{\partial F_1}{\partial\alpha_1} = 0. \tag{55c}$$

The solutions are the same as (51c) and (51b). Thus when the rotation is strong enough the solutions of the Euler-Lagrange equations of the variational problem based on the optimum fields with and without Ekman layer lead to the same thickness of the boundary layer of the optimum field and as a consequence to the same upper bound on the convective heat transport.

4.4 Case 4: Boundary layer is thicker than the Ekman layer and $O(R^{1/4}) \gg O(Ta^{1/6}) \gg \alpha_1 \gg O\left(\left(\frac{Ta}{R}\right)^{1/2}\right) \gg O(R^{1/8})$

The solution of the Euler-Lagrange equations of the variational problem for w_1 in the intermediate layer is

$$w_1 = \frac{z}{\alpha_1\kappa}\sqrt{\ln\left(\frac{1}{z^2}\right)-\ln\ln\left(\frac{1}{z^2}\right)}, \tag{56}$$

where $\kappa = Ta/\alpha_1^6$ is a large quantity. We match the solutions between the intermediate and boundary layers (Ekman layer is very thin) and obtain the relationships: $w_1 = A\eta$; $\theta_1 = \eta/(A(1+\eta^2))$. $\eta = z/\delta$ is the coordinate in the boundary layer, δ can be connected to the thickness of the boundary layer of the optimum field. The relationship for A is

$$A = \frac{\alpha_1^2 \delta}{Ta^{1/2}} \sqrt{\ln\left(\frac{1}{\delta^2}\right) - \ln\ln\left(\frac{1}{\delta^2}\right)}. \tag{57}$$

For the upper bounds on the convective heat transport and the corresponding thickness of the boundary layer of the optimum field we have to solve the equations

$$F_1 = \frac{2(1-\alpha_1^4/R)}{\pi(\delta + 1/(2\delta\alpha_1^2))} \tag{58a}$$

$$\frac{RF_1\alpha_1^2\delta^2}{Ta}\left[\ln\left(\frac{1}{\delta^2}\right) - \ln\ln\left(\frac{1}{\delta^2}\right)\right] = 1. \tag{58b}$$

The solutions are

$$F = \frac{4R\alpha_1^2}{\pi^2 Ta}\left[\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right) - \ln\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right)\right], \tag{59a}$$

$$\delta_{\rm u} \propto \delta_{\rm l} = \frac{\pi Ta}{2R\alpha_1^2} \times \left[\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right) - \ln\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right)\right]^{-1}. \tag{59b}$$

5 Fluid layer with two rigid boundaries

5.1 Case 1: Ekman layer is thicker than the boundary layer and $O(Ta^{1/8}) \ll \alpha_1 \ll O(R^{1/4})$

The rotation effects are important in the internal layers of the optimum field and the intermediate layers expand in the direction of the internal layers. We assume that $w_1\theta_1 \approx 1$ holds in the intermediate layer of the optimum fields. Assuming that the terms in the Euler-Lagrange equations which contain derivatives and do not contain the Taylor number are much smaller than the other terms we obtain the equations (5a–c) which can be further reduced to the equation (6) possessing solution (7a). Assuming further that $w_1\theta_1 \approx 1$ holds also in the Ekman layers we obtain the equations (8a, 8b) plus corresponding boundary conditions. The matching of the solutions of the Euler-Lagrange equations between the intermediate and Ekman layers lead to the relationships (11a) for z_0 and (11b) for c. In the boundary layer the significant terms in the Euler-Lagrange equations are these ones which contain the highest derivatives. Thus the Euler-Lagrange equations are reduced to (14a–c). Introducing the boundary-layer coordinate $\eta = z/\delta$ where δ can be connected to the thickness of the boundary layer of the field $w_1\theta_1$ we obtain the following solutions of the Euler-Lagrange equations in the boundary layer

$$w_1 = \frac{1}{\sqrt{2}} c Ta^{1/2}\delta\eta^2, \tag{60a}$$

$$f_1 = \sqrt{2} c Ta^{1/2}\delta\eta. \tag{60b}$$

The equations we have to solve in order to obtain expressions for the upper bound on the convective heat transport, for the thickness of the boundary layer δ, and for the wave number α_1 corresponding to the optimum fields are

$$F_1 = \frac{1-\alpha_1^4/R}{2\delta D}, \tag{61a}$$

$$RF_1\delta^6 c^2 Ta = 2, \tag{61b}$$

$$c = \frac{1}{2Ta^{1/4}}\sqrt{\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right) - \ln\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right)}. \tag{61c}$$

where $D = I + J$,and I and J are the same as $I_{\rm l}$ and $J_{\rm l}$ from (18d) and (18f). The solutions are

$$\delta = 2^{4/5}D^{1/5}(R-\alpha_1^4)^{-1/5}Ta^{-1/10} \times \left[\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right) - \ln\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right)\right]^{-1/5}, \tag{62a}$$

$$F_1 = 2^{-9/5}D^{-6/5}R^{1/5}(1-\alpha_1^4/R)^{6/5}Ta^{1/10} \times \left[\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right) - \ln\ln\left(\frac{2\alpha_1^4}{Ta^{1/2}}\right)\right]^{1/5}. \tag{62b}$$

We cannot directly optimise F_1 with respect to α_1 because this will move the integral of the validity of the obtained solution in an area of Rayleigh and Taylor numbers in which no convective motion is possible *i.e.* we shall come to a contradiction with the assumption that we have thermal convection in the investigated fluid layer. In order to obtain a maximising expression for the convective heat transport we take into an account that F_1 in (62b) increases with increasing α_1 but α_1^4/R must be smaller than 1. Thus a maximum of the convective heat transport in the interval

$$O(Ta^{1/8}) \ll \alpha_1 \le O(R^{1/4}), \tag{63}$$

can be obtained by the assumption $\alpha_1 \propto Ta^{1/6}$ which leads us to the following relationships for the convective heat transport F_1 and for the boundary layer thickness δ

$$\delta = 2^{4/5}D^{1/5}R^{-1/5}Ta^{-1/10} \times [\ln(2Ta^{1/6}) - \ln\ln(2Ta^{1/6})]^{-1/5}, \tag{64a}$$

$$F_1 = 2^{-9/5}D^{-6/5}R^{1/5}Ta^{1/10} \times [\ln(2Ta^{1/6}) - \ln\ln(2Ta^{1/6})]^{1/5}. \tag{64b}$$

5.2 Case 2: Ekman layer is thicker than the boundary layer and $O((Ta/R)^{1/2}) \ll \alpha_1 \ll O(Ta^{1/8})$

We assume that $w_1\theta_1 \approx 1$ overall except in the Ekman and boundary layers of the optimum field. The solutions of the Euler-Lagrange equations in the intermediate layer are $w_1 = w_0/\alpha_1$; $\theta_1 = \alpha_1/w_0$, where w_0 must be a small constant as well as $\alpha_1^2 f_1 \ll 1$. The solution for f_1 is

$$f_1 = \alpha_1^3(1/2 - z)/(w_0 Ta).$$

The solutions for the Ekman layer are (9a) and (9b) (where we put c and ϕinstead of c_l and ϕ_l) with Ekman layer coordinate $\phi = Ta^{1/4}z/\sqrt{2}$. c is a constant which together with w_0 has to be determined from the matching of the solutions of the Euler-Lagrange equations of the variational problem between the intermediate and Ekman layers of the optimum field. The results of this matching are:

$$w_0 = 2^{-3/4}\alpha_1^2 Ta^{-3/8};\ c = 2^{-5/4}\alpha_1 Ta^{-3/8}$$

and because of the assumption $\alpha_1 \ll O(Ta^{1/8})$ under which we discuss this case the quantities $w_0 \ll 1/Ta^{1/8}$ and $\alpha_1^2 f_1 \ll 1/Ta^{1/4}$ are small indeed.From the Euler-Lagrange equation for the boundary layer of the optimum field we obtain the relationship

$$RF_1\delta^2 c^2 Ta = 2. \tag{65a}$$

For F_1 we have

$$F_1 \frac{1 - \alpha_1^4/R}{2\delta D}, \tag{65b}$$

where $D = I_l + J_l$. I_l, J_l are the same as (18d) and (18f). (65a) and (65b) lead to the following solutions for δ and F_1

$$\delta = 2^{1/2}(4D)^{1/5}R^{-1/5}Ta^{-1/20}\alpha_1^{-2/5}(1 - \alpha_1^4/R)^{-1/5}, \tag{66a}$$

$$F = 2^{-19/10}(D)^{-6/5}R^{1/5}Ta^{1/20}\alpha_1^{2/5}(1 - \alpha_1^4/R)^{6/5}. \tag{66b}$$

As F_1 increases with increasing α_1 its maximum value can be obtained when $\alpha_1 \to O(Ta^{1/8})$. The result is

$$F_1 \propto 2^{-19/10}D^{-6/5}R^{1/5}Ta^{1/10}, \tag{67a}$$

$$\delta \propto 2^{1/2}(4D)^{1/5}R^{-1/5}Ta^{-1/10}. \tag{67b}$$

Thus we obtain the same power law dependencies for δ and F_1 as in the case for a fluid layer with rigid boundaries discussed above (except the corresponding logarithmic corrections).

5.3 Case 3: Boundary layer is thicker than the Ekman layer and $O((R \ln R)^{4/3}) \ll Ta \ll O(R^{3/2})$

The boundary layer becomes thicker than the Ekman layer. The rotation leads to a merging of the internal and intermediate layers of the optimum fields. In the intermediate layer we have $w_1\theta_1 \approx 1$ in addition to the equations (5a) and (5b). Introducing the coordinate $\xi = \alpha_1^3 z/Ta^{1/2}$ we obtain the solution for w_1 in this layer

$$w_1 = \frac{\xi}{\alpha_1}\sqrt{\ln\left(\frac{1}{\xi^2}\right) - \ln\ \ln\left(\frac{1}{\xi^2}\right)}. \tag{68}$$

In the Ekman layer $w_1\theta_1 \approx 1$ holds further. The Euler-Lagrange equations are (8a) and (8b) and the solutions for w_1 and f_1 are: $w_1 = c\sqrt{2} - 2c e^{-\phi}\cos(\phi - \pi/4)$; $f_1 = Ta^{1/4}(2c - 2c e^{-\phi}\cos(\phi))$. $\phi = Ta^{1/4}z/\sqrt{2}$ is the coordinate in the Ekman layer and c is an appropriate constant of integration. Let us introduce the boundary layer coordinate $\eta = z/\delta$ (δ can be connected to the thickness of the boundary layers of the optimum field $w_1\theta_1$). A matching between the intermediate and the boundary layer leads to the following relationships for $w_1 = A\hat{w}_1$ in the boundary layer

$$A = \frac{\alpha_1^2}{Ta^{1/2}}\sqrt{\ln\left(\frac{Ta}{\alpha_1^6\delta^2}\right) - \ln\left(\frac{Ta}{\alpha_1^6\delta^2}\right)}, \tag{69}$$

and $\hat{w}_1 = \eta$ that holds in almost the whole boundary layer except for a small region near the fluid boundary where $\hat{w}_1$ changes in order to satisfy the boundary conditions. We assume that these changes are negligible from the point of view of the integrals we have to calculate in order to obtain the upper bound on the convective heat transport. In the boundary layer the terms in the Euler-Lagrange equations which contain derivatives and do not contain the Taylor number are small in comparison to the other terms in the Euler-Lagrange equation. Thus we obtain the following equation for $\theta_1 = \hat{\theta}_1/A$

$$\frac{\alpha_1^2}{RF_1A^2}\hat{\theta}_1 = \hat{w}_1(1 - \hat{w}_1\hat{\theta}_1). \tag{70}$$

Assuming further that $\alpha_1^2/(RF_1A^2) = 1$ the solution for $\hat{\theta}_1$ is

$$\hat{\theta}_1 = \frac{\eta}{1 + \eta^2}, \tag{71}$$

which satisfies the corresponding boundary condition on the lower boundary of the fluid layer.

The equations for the upper bound on the convective heat transport as well for the corresponding wave number and thickness of the boundary layers are

$$F_1 = 2\frac{1-(\alpha_1^4+\pi/(4\delta A^2))/R}{(\pi\delta)}, \tag{72a}$$

$$Ta = \alpha_1^2\delta^2 RF_1\left[\ln\left(\frac{Ta}{\alpha_1^6\delta^2}\right)-\ln\left(\frac{Ta}{\alpha_1^6\delta^2}\right)\right]^{-1}, \tag{72b}$$

$$\frac{\partial F_1}{\partial\alpha_1}=0. \tag{72c}$$

Their solutions are

$$\alpha_1=\left(\frac{R}{5}\right)^{1/4}, \tag{73a}$$

$$F_1=\frac{64R^{3/2}}{25\sqrt{5}\pi^2 Ta}\times\left[\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)-\ln\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)\right], \tag{73b}$$

$$\delta=\frac{5\sqrt{5}\pi Ta}{8R^{3/2}}\times\left[\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)-\ln\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)\right]^{-1}. \tag{73c}$$

5.4 Case 4: Boundary layer is thicker than the Ekman layer and $\mathbf{O(R^{1/8}) \ll O((Ta/R)^{1/2}) \ll \alpha_1 \ll O(Ta^{1/6}) \ll O(R^{1/4})}$

When Ta increases further $\kappa^2 = Ta/\alpha_1^6$ is a large quantity. Equations (5a) and (5b) lead to the following solution for w_1 in the intermediate layer

$$w_1=\frac{z+z_0}{\alpha_1\kappa}\times\sqrt{\ln\left(\frac{1}{(z+z_0)^2}\right)-\ln\ln\left(\frac{1}{(z+z_0)^2}\right)}. \tag{74}$$

In the Euler-Lagrange equations in the boundary layer the terms that contain derivatives and do not contain the Taylor number are negligible. Thus the equations for w_1 and θ_1 are

$$-\alpha_1^6 w - 1 + Ta\frac{\mathrm{d}^2 w_1}{\mathrm{d}z^2}+\alpha_1^4\theta_1 = 0 \tag{75a}$$

$$-\frac{\alpha_1^2 Ta}{RF_1}\frac{\mathrm{d}^2\theta_1}{\mathrm{d}z^2}+Ta\frac{\mathrm{d}^2}{\mathrm{d}z^2}[w_1(1-w_1\theta_1)]=0. \tag{75b}$$

Introducing the boundary layer coordinate $\eta = z/\delta$ and solving the Euler-Lagrange equations we obtain

$$w_1 = A\eta, \tag{76a}$$

$$A=\frac{\alpha_1^2\delta}{Ta^{1/2}}\sqrt{\ln\left(\frac{1}{\delta^2}\right)-\ln\ln\left(\frac{1}{\delta^2}\right)}, \tag{76b}$$

$$\theta_1=\frac{1}{A}\frac{\eta}{1+\eta^2}, \tag{76c}$$

$$\frac{\alpha_1^2}{RF_1A^2}=1, \tag{76d}$$

$$F_1=\frac{2(1-\alpha_1^4/R)}{\pi(\delta+1/(2\delta\alpha_1^2))}\cdot \tag{76e}$$

The last two equations lead to the upper bound

$$F=\frac{4R\alpha_1^2}{\pi^2 Ta}\left[\ln\left(\frac{4R^2\alpha_1^4}{\pi^2Ta^2}\right)-\ln\ln\left(\frac{4R^2\alpha_1^4}{\pi^2Ta^2}\right)\right], \tag{77a}$$

and to corresponding value of δ:

$$\delta=\frac{\pi Ta}{2R\alpha_1^2}\left[\ln\left(\frac{4R^2\alpha_1^4}{\pi^2Ta^2}\right)-\ln\ln\left(\frac{4R^2\alpha_1^4}{\pi^2Ta^2}\right)\right]^{-1}. \tag{77b}$$

6 Intervals of validity of the obtained bounds

For the case 1 the constants z_0 and z_0^* must be small. Moreover F_1 must be large and positive. Thus $\alpha_1 \ll O(R^{1/4})$. Remembering that we consider the interval of large Taylor numbers $O(Ta^{1/6}) \gg \alpha_1$ we have the following interval of the application of the obtained solutions

$$O(Ta^{1/8}) \ll \alpha_1 \ll O(Ta^{1/6}) < O(R^{1/4}). \tag{78}$$

For case 2 and rigid-stress-free boundaries the interval of the validity of the obtained bounds is determined by the requirements that we discuss the case $\alpha_1 \ll O(Ta^{1/8})$, that the convective heat transport for the discussed case must be greater than the convective heat transport in the corresponding case where the boundary layers of the optimum field are thicker than the Ekman layers, and finally the the boundary layers of the optimum field must be thin. Thus we obtain

$$O((Ta/R)^{1/2}) \ll \alpha_1 \ll O(Ta^{1/8}) \ll O(R^{3/14}). \tag{79}$$

For the other two kinds of boundary conditions the same requirements lead to the interval of validity $O((Ta/R)^{1/2}) \ll \alpha_1 \ll O(Ta^{1/8})$.

For the case 3 the requirements for the region of validity of the obtained bounds are two: large value of the convective heat transport, from which follows $O(Ta) \ll O(R^{3/2})$, and small size of the boundary layer which leads to $O(Ta) \gg O((R\ln R)^{4/3})$. For the intervals

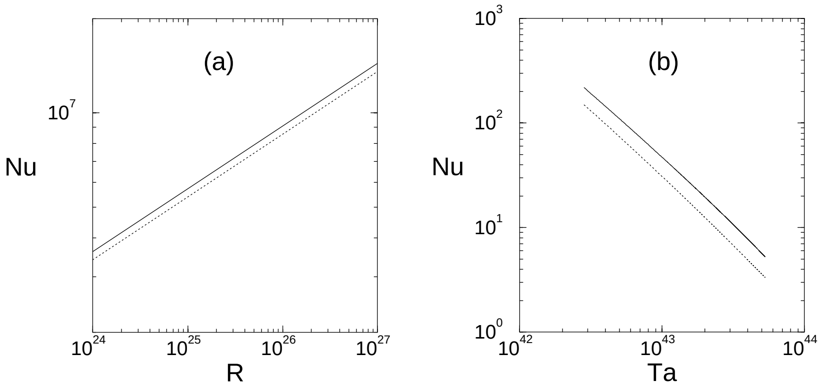

Fig. 3. Comparison of the bounds on the convective heat transport. Panel (a): Fluid layer with rigid boundaries, case 1. Solid line: upper bound obtained by Hunter and Riahi [7]. Dotted line: upper bound obtained in this article. Panel (b): Fluid layer with stress-free boundaries, case 3. Solid line: upper bound obtained by Chan [6]. Dotted line: upper bound obtained in this article.

of the validity of the obtained bound in the case 4 we note the following. We discuss the interval $\alpha_1 \ll O(Ta^{1/6})$ and from equation (35a) it is clear that $\alpha_1 < O(R^{1/4})$. From (36a) we must have also $\alpha_1 \gg O((Ta/R)^{1/2})$ and finally $\alpha_1 \gg O(R^{1/8})$. Thus the interval of validity of the bounds is

$$O(R^{1/4}) \gg O(Ta^{1/6}) \gg \alpha_1 \gg O\left(\left(\frac{Ta}{R}\right)^{1/2}\right) \gg O(R^{1/8}). \qquad (80)$$

As a motivation for such a interval we note that for the fluid layer with two stress-free boundaries as well as for the fluid layer with two rigid boundaries and with rigid lower boundary and stress-free upper boundary we have from the linear stability theory that for the case when the instability is set in as stationary convection we obtain the asymptotic relationships for the critical Rayleigh number as well as for the corresponding wave number for the case $Ta \to \infty : R_c \propto O(Ta^{2/3})$ and $\alpha_c \propto O(Ta^{1/6})$ [51].

7 Discussion

In this article we have obtained upper bounds on the convective heat transport in a fluid layer heated from below and rotating about a vertical axis. The bounds are derived on the basis of asymptotic solutions of the Euler-Lagrange equations for the corresponding variational problem, obtained by application of the Howard-Busse method of the optimum theory of turbulence to the Boussinesq equations. We have shown that for the case of a fluid layer with stress-free boundaries the Euler-Lagrange equations of the variational problem possess solutions based on an optimum field with four layers (Ekman layer inclusive). The coexistence of three layers solution and four layers solution of the Euler-Lagrange equations is a consequence of their nonlinearity.

In the case without rotation [24] the thickness of the boundary layer of the optimum field for the rigid boundary is larger than the thickness of the boundary layer corresponding to the stress-free boundary. Thus the upper bound on the convective heat transport for the case of rigid lower boundary and stress-free upper boundary lies between the bound for the cases of rigid boundaries and stress-free boundaries. For the case with rotation the assumptions that one of the two boundary layers is much thicker than the other one conflicts with the other assumptions made in the process of the developing of the asymptotic theory. The only assumption which does not lead to a conflict with the other assumptions is that the thickness of both boundary layers is of the same order. The difference between this assumption and the assumption that the rigid boundary layer is much thicker than the stress-free boundary layer is in the terms $D_{\mathrm{u,l}}$. If we assume that the boundary layer thicknesses are of the same order, we obtain the term $D_{\mathrm{u}} + D_{\mathrm{l}}$ in the relationships for the upper bound, wave number and thicknesses of the boundary layers (the assumption that the rigid boundary layer is much thicker than the stress-free boundary layer of the corresponding optimum field leads to the term D_{l} instead of $D_{\mathrm{u}} + D_{\mathrm{l}}$).

When the Taylor and Rayleigh numbers are high enough the boundary layers of the optimum fields for rigid, stress-free and rigid-stress free boundary conditions have the same thickness. This leads to the same expressions for the convective heat transport for all three cases.

Hunter and Riahi [7] have discussed the case of rigid boundaries and their results can be compared to the results of Section 4 of this article. The results of Chan [6] can be compared to the results of case 3 of Section 3. For the case of rigid boundary conditions the boundary layer of the optimum field for the solution used above is thicker than the corresponding boundary layer of the optimum field for the solution of Hunter and Riahi. Thus the value of the convective heat transport obtained here is smaller than the value obtained in [7] for fixed Rayleigh and Taylor numbers. For case 1 the difference between the bound obtained here and the bound of Hunter and Riahi depends only on the Taylor number. Case 2 leads to the same results as in [7] and we present it as briefly as possible. In the cases 3 and 4 the difference depend on both Rayleigh and Taylor numbers. The comparison is shown in Figure 3. For panel (a) the Taylor number has the fixed

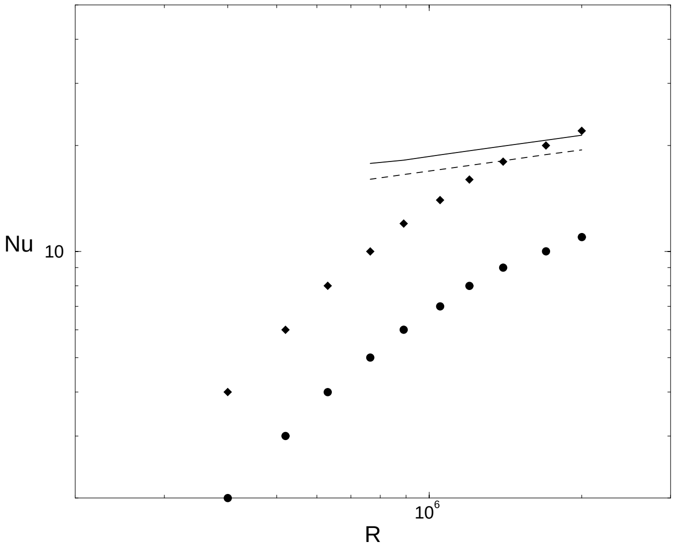

Fig. 4. Comparison of the bounds on the convective heat transport to the experimental results of Rossby. Circles: Experimental results of Rossby for water at $Ta = 10^6$. Diamonds: Results of Rossby multiplied by 2. Solid line: upper bound of Hunter and Riahi. Dashed line: upper bound obtained in this article.

value $Ta = 10^{20}$. For panel (b) the Rayleigh number is fixed: $R = 10^{40}$. This panel shows that the solution for intermediate layers of the optimum fields leads to correction of the upper bound.

The obtained results can be compared to the experimental results of Rossby [52]. The comparison is shown in Figure 4. The bounds obtained by Hunter and Riahi can be treated as upper bounds on the obtained in this article upper bounds. For the case of the comparison to the experimental results the bound (64b) is used. When $Ta = 10^6$ we have the value $F_1 \approx 1.15R^{1/5}$. The corresponding bound of Hunter and Riahi is obtained when the doubly logarithmic term in (64b) is omitted. The result is $F_1 \approx 1.26R^{1/5}$ *i.e.* the Hunter and Riahi bound is about 10% higher than the bound obtained in this article.

The comparison above is between experimental data and asymptotic bounds. This explains the difference for small values of Rayleigh number. With increasing Rayleigh number the difference decreases but nevertheless the experimental values of Rayleigh number are far from the asymptotic region where the assumptions made for obtaining the upper bound hold satisfactory enough. The comparison shows a tendency that with increasing Rayleigh number the upper bounds come close to the experimental data: the ratio between the experimental data and the bound becomes smaller than 2.

The bounds obtained in this article can be compared to the bounds obtained in [49] and [50]. We note that

– An useful feature of the Doering-Constantin method is that every background field that satisfies the corresponding conditions, leads quickly to an upper bound on the investigated quantity. Among the bounds that can be obtained by the admissible background fields we can find the lowest one. This bound should be not higher than the bound obtained by means of the Howard-Busse method.
– The bounds, obtained in [49,50] have larger application area with respect to the Rayleigh and Taylor numbers in comparison to the bounds, obtained in this article. As we shall see, the bounds obtained here are lower but we must keep in the mind that these bounds are obtained on the basis of the $1-\alpha$-solution of the variational problem. The bounds, obtained by the multi-α-solutions of the variational problem are higher than the bounds, obtained by the$1-\alpha$-solution of the variational problem. It is well known that each of these bounds has its area of application and can be compared to the Doering-Constantin bounds only in this area. One future direction of investigations will be the problem of the construction of a theory of the multi-α bounds and comparison of the obtained bounds to these of Doering and Constantin in order to see whether all of the multi-α-bounds will be lower than the Doering-Constantin bound or some of them will be higher.
– For the intervals of Rayleigh and Taylor numbers, discussed in this article, the obtained bounds must be compared to the following upper bound obtained by means of the Doering-Constantin method

$$Nu \leq 0.6635R^{2/5}. \tag{81}$$

We note that the background field used for obtaining this bound has a minimum layer structure: two boundary layers and internal layer. The optimum fields, discussed in this article have more complex structure, that includes also intermediate layers and Ekman layers. One can consider the inclusion of more layers in the optimum fields as imposing additional constraints on the variational problem, that lead to further decreasing of the manifold of the fields among which the optimal ones are searched.

In order to compare the obtained in this article bounds to the bound (81) we have to express our bounds as a function of the Rayleigh number. We shall express all quantities in the equations for the bounds by means of the Rayleigh number. We shall put the maximum possible value of the corresponding quantities if they are in the numerator and the minimum possible values of the quantities that are in the denominator. Thus we shall obtain an upper bound on the corresponding upper bound and this higher upper bound will be a function only of the Rayleigh number. The result for the case 1 for a fluid layer with rigid boundaries is

$$Nu < R^{1/3}[\ln(2R^{2/9}) - \ln\ln(2R^{2/9})]^{1/5}. \tag{82a}$$

The result for the case 1 of a fluid with stress-free boundaries is

$$Nu < R^{1/3}[\ln(R^{1/3}/2) - \ln\ln(R^{1/3}/2)]^{1/5}. \qquad (82b)$$

For the cases 2–4 the results are the same for the two kinds of boundary conditions. Thus for:

– Case 2:

$$Nu < R^{1/3} \qquad (82c)$$

– Case 3: Assuming that $\ln(R^{4/3}) > 1$ we obtain

$$Nu < R^{1/6}[\ln(R^{1/6}) - \ln\ln(R^{1/6})] \qquad (82d)$$

– Case 4:

$$Nu < R^{1/4}[\ln(R^{1/2}) - \ln\ln(R^{1/2})]. \qquad (82e)$$

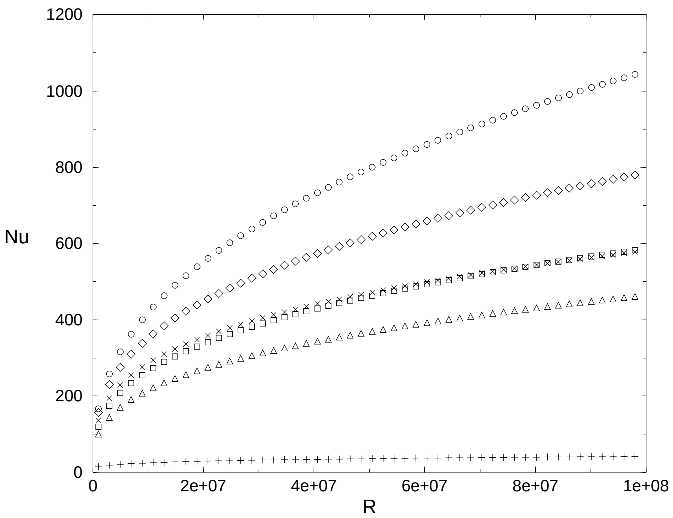

Fig. 5. Comparison of the right-hand sides of the inequalities (81) and (82) for interval of Rayleigh numbers between 10^6 and 10^8. Circles: the bound (81). Squares: the bound (82a). Diamonds: the bound (82b). Triangles: the bound (82c). +: the bound (82d). ×: the bound (82e).

In order to get an impression of the development of the right-hand sides of (81) and (82) with increasing Rayleigh number we present them in Figure 5. The Euler-Lagrange equations obtained by the Howard-Busse method can be obtained also by means of the Doering-Constantin method. Thus Figure 5 is a visualisation of the size of the effect that the inclusion of more assumptions for the layer structure of the optimum fields can have on the upper bounds on the convective heat transport.

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